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Rectangle formula with randomly shifted knots

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Abstracts. The classical rectangle formula of calculating integrals is modified to be applicable to L_2-functions by using the method of approximation by families of linear operators. The algorithm of numerical integration is developed according to the modified formula and its approximation properties dependent on the input parameters and function to be integrated are evaluated

Keywords: families of linear operators, piecewise constant functions, cubature formulas, modulus of continuity, stochastic approximation

Introduction

In this paper we study the classical rectangle formula with randomly shifted knots (see e. g. [2]) and we show that such a method is relevant for the numerical integration of square-integrable functions. Our approach is based on applying the method of approximation by families of piecewise constant functions systematically studied in [1]. More precisely, we discuss the following formula

$$\int_0^1 f(x)\,dx \quad \sim \quad \frac{1}{n}\sum_{k=0}^{n-1} f^*\left(\frac{k}{n}+\lambda\right)\,,\ f^*\in L_2\,,\ n\in\mathbb{N}\,, \tag{0.1}$$

where L_2 is the space of 1-periodic square-integrable functions equipped with the standard norm, f^* is a periodic extension of f and λ is a uniformly distributed on $[\,0,1)$ random variable. For the sake of simplicity we denote the left-hand side and the right-hand side of (0.1) by $I(f)$ and $I_{n;\lambda}(f)$, respectively.

By the symbol $\mathcal{C}_n$ we denote the set

$$\mathcal{C}_n = \bigcup_{\tau\in\mathbb{R}} \mathcal{C}_{n,\tau}\,,$$

where $\mathcal{C}_{n,\tau}$, $\tau\in\mathbb{R}$, is the space of 1-periodic functions $f(x)$ satysfying $f(x+\tau)=c_k$ for $x\in\left[\,\frac{k}{n},\frac{k+1}{n}\right)$, $k=0,1,\ldots,n-1$. Clearly, $\mathcal{C}_{n,\lambda}=\mathcal{C}_{n,\lambda+\frac{1}{n}}$. Following [1], we define the family of operators mapping L_2 into the space $\mathcal{C}_n$, $n\in\mathbb{N}$ of piecewise constant functions by (λ is a real parameter)

$$\mathcal{I}_{n;\lambda}(f;x)=\frac{1}{n}\sum_{k=0}^{n-1} f\Big(\frac{k}{n}+\lambda\Big)\,\mathcal{K}_n\Big(x-\frac{k}{n}-\lambda\Big)\,,\ n\in\mathbb{N}\,, \tag{0.2}$$

where the 1-periodic function $\mathcal{K}_n$ given on the period by setting

$$\mathcal{K}_n(h) = \begin{cases} n, h \in \left[0, \dfrac{1}{n}\right) \\ 0, h \in \left[\dfrac{1}{n}, 1\right) \end{cases} \tag{0.3}$$

is called kernel.

Dealing with families (0.2) we consider functions in $L_2([0,1)^2)$, which depending on two variables x and λ. We use the symbol $\| \cdot \|_{[2]}$ to denote the corresponding norm, i. e.,

$$\| f \|_{[2]} = \Big(\int_0^1 \int_0^1 | f(x, \lambda) |^2 \, dx d\lambda \Big)^{1/2} . \tag{0.4}$$

As it was shown in [1], the quality of approximation by families (0.2) can be described in terms of the modulus of continuity given by

$$\omega(f, \delta) = \sup_{0 \le h \le \delta} \| f^*(x+h) - f^*(x) \|_2 \, , \; 0 \le \delta \le 1 \, , \tag{0.5}$$

More precisely, the following inequality holds.

$$\| f - \mathcal{I}_{n;\lambda}(f) \|_{[2]} \le 9 \, \omega(f, n^{-1}) \, , \; f \in L_2 \, , \; n \in \mathbb{N} \, , \tag{0.6}$$

The paper is organized as follows. The main result on quality of formula (0.1) is formulated and proved in Section 1. Section 2 is devoted to description of the corresponding algorithm of numerical integration. In Section 3 the table demonstrating the approximating properties of the algorithm is placed.

1. Main result

In this section $P\{A\}$ denotes the probability of an event A. The symbols $\mathcal{E}$ and $\mathcal{D}$ stand for mathematical expectation and variance, respectively.

Theorem. *Let $m \in \mathbb{N}$ and η_j, $j = 1, \ldots, m$, be independent random variables uniformly distributed on $[0,1]$, $\theta_j = \eta_j / n, j = 1, \ldots, m$. Then for $f \in L_2$, $n \in \mathbb{N}$ and $\varepsilon > 9m^{-1/2}\omega(f, n^{-1})$*

$$P\Big\{ \Big| \frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) - I(f) \Big| < \varepsilon \Big\} \ge 1 - \Big(\frac{9\,\omega(f, n^{-1})}{\sqrt{m}\,\varepsilon} \Big)^2 . \tag{1.1}$$

Proof. Taking into account that the function $I_{n;\lambda}(f)$ is n^{-1}-periodic and

$$dF_{\theta_j}(\lambda) = n \, d\lambda \, , \; \lambda \in [0, \frac{1}{n}) \, , \tag{1.2}$$

for $j = 1, \ldots, m$, where F_{θ_j} is the distribution function of θ_j, we get by the definitions of expectation and variance

$$\mathcal{E}\left(I_{n;\theta_j}(f) \right) = \mathcal{E}\left(I_{n;\theta_1}(f) \right) = n \int_0^{\frac{1}{n}} I_{n;\lambda}(f)\, d\lambda = \int_0^1 I_{n;\lambda}(f)\, d\lambda = I(f)\,, \tag{1.3}$$

$$\mathcal{D}\left(I_{n;\theta_j}(f) \right) = \mathcal{D}\left(I_{n;\theta_1}(f) \right) \;=\; \mathcal{E}\Big(\left(I_{n;\theta_1}(f) - \mathcal{E}\left(I_{n;\theta_1}(f) \right) \right)^2 \Big) =$$

$$= n \int_0^{\frac{1}{n}} \Big(I_{n;\lambda}(f) - n \int_0^{\frac{1}{n}} I_{n;t}(f)\, dt \Big)^2 d\lambda = \int_0^1 \left| I_{n;\lambda}(f) - I(f) \right|^2 d\lambda\ . \tag{1.4}$$

In view of the Chebyshev inequality and (1.3) the left-hand side of (1.1) can be estimated from below by

$$1 \,-\, \varepsilon^{-2}\, \mathcal{D}\Big(\frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) \Big)\,. \tag{1.5}$$

As the values $I_{n;\theta_j}$ are independent and they have one and the same distribution law, we get

$$\mathcal{D}\Big(\frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) \Big) \;=\; m^{-2} \sum_{j=1}^{m} \mathcal{D}\big(I_{n;\theta_j}(f) \big) \;=\; m^{-1}\, \mathcal{D}\big(I_{n;\theta_1}(f) \big)\,. \tag{1.6}$$

Combining (1.4)-(1.6) we get

$$P\Big\{ \Big| \frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) - I(f) \Big| \,<\, \varepsilon \Big\} \;\geq\; 1 \,-\, \varepsilon^{-2}\, m^{-1} \int_0^1 \Big| I_{n;\lambda}(f) - I(f) \Big|^2 d\lambda\ . \tag{1.7}$$

Applying (0.6) in combination with Hölder inequality and taking into account that

$$\int_0^1 \mathcal{I}_{n;\lambda}(f, x)\, dx \;=\; I_{n;\lambda}(f)\ , \tag{1.8}$$

we get

$$\int_0^1 \Big| I(f) - I_{n;\lambda}(f) \Big|^2 d\lambda \leq \int_0^1 \Big(\int_0^1 \left| f(x) - \mathcal{I}_{n;\lambda}(f, x) \right| dx \Big)^2 d\lambda \tag{1.9}$$

and

$$\int_0^1 \Big(\int_0^1 \left| f(x) - \mathcal{I}_{n;\lambda}(f, x) \right| dx \Big)^2 d\lambda \leq \| f - \mathcal{I}_{n;\lambda}(f) \|_{[2]}^2 \;\leq\; 81\, \omega(f, n^{-1})^2\ . \tag{1.10}$$

That means

$$P\Big\{ \Big| \frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) - I(f) \Big| < \varepsilon \Big\} \ \geq \ 1 \ - \ 81\,\varepsilon^{-2}\, m^{-1}\, \omega(f, n^{-1})^2 \ =$$

$$= 1 \ - \ \Big(\frac{9\,\omega(f, n^{-1})}{\sqrt{m}\,\varepsilon} \Big)^2 . \qquad (1.11)$$

The proof is complete. □

2. Computational aspects

In this section we describe the algorithm of numerical integration of L_2-functions based on Theorem above. It has the following input parameters:

(i) $f \in L_2$ — function to be integrated;

(ii) $\varepsilon \in (0, +\infty)$ — approximation error;

(iii) $\sigma \in (0, 1)$ — probability error;

(iv) $n \in \mathbb{N}$ — number of knots or volume of the grid;

(v) $m \in \mathbb{N}$ — number of random shifts of the grid.

The parameters ε, σ, n and m are dependent on each other. If the approximation error and probability error are given, then we are looking for n and m satisfying

$$P\Big\{ \Big| \frac{1}{m} \sum_{j=1}^{m} I_{n;\theta_j}(f) - I(f) \Big| < \varepsilon \Big\} \ \geq \ 1 \ - \ \sigma \ . \qquad (2.1)$$

If the estimate from above for the modulus of continuity is known, i. e., $\omega(f, \delta) \leq \omega(\delta)$ for $0 \leq \delta \leq 1$, in view of (1.1) it is sufficient to require

$$c_n \ \equiv \ 9\,\omega(n^{-1}) \ \leq \ \sqrt{m\sigma}\,\varepsilon \qquad (2.2)$$

in order to achieve the desired level of both the approximation and probability errors.

The computational procedure can be described as follows. First we choose independent uniformly distributed on $[0, 1)$ random variables η_j, $j = 1, \ldots, m$, and we determine $\theta_j = \eta_j / n$, $j = 1, \ldots, m$. After that we compute the numbers

$$I_{n;\theta_j}(f) \ = \ \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n} + \theta_j \right) \ , \quad (\, j = 1, \ldots, m \,) \ . \qquad (2.3)$$

Finally we calculate

$$I_n^*(f; \Phi) \ = \ \frac{1}{m} \sum_{j=1}^{m-1} I_{n;\theta_j}(f) \ , \quad (\, \Phi \ = \ \{\, \theta_j,\, j = 1, \ldots, m \,\}\,) \ . \qquad (2.4)$$

By Theorem and (2.2) this number satisfies (2.1), i. e., it approximates $I(f)$ up to ε with the probability at least $1-\sigma$.

Relation (2.2) can be fulfilled in some different ways. If n is given, its left-hand side c_n is a constant and the prescribed level of errors is achieved by choosing the parameter m only. Taking into account that it should be chosen as small as possible we get from (2.2) the explicit formula for its optimal value

$$m = \left[\frac{c_n^2}{\sigma\varepsilon^2} \right] + 1 . \tag{2.5}$$

It is interesting to notice that the classical Monte-Carlo method is a special case of our method corresponding to $n = 1$. It is convenient, if we lack information on the smoothness properties of a given function f or f is "bad" in the sense that its modulus of continuity tends to 0 slowly. On the other hand, m is the only parameter which is responsible for both errors. Thus, the Monte-Carlo method being quite universal does not take into account any information on the smoothness properties of a function to be approximated.

If we a priori determine m, the necessary level of errors can be reached by increasing the parameter n. It is an effective and economic way of calculations, if a given function has a"good" smoothness in L_2. Indeed, in contrast to the Monte-Carlo method it is not necessary now to increase the number of random variables, in order to achieve the desired approximation error, that is, we do not need to be "too careful" when choosing the generator of random variables we would like to use. Reminding the well-known principle that the greater m, the "better" generator should be, we avoid in such a way one of the most complicated technical problems, which does always arise, if stochastic approaches are applied.

In a certain sense, our method with fixed m is close to the classical rectangle formula, which can be interpreted as the degenerated case of the general situation corresponding formally to $m = 0$. On the other hand, its properties are much better than the properties of classical cubatures. Indeed, in contrast to the rectangle formula with fixed nodes, which is applicable only for continuous functions, we can integrate functions in L_2, even, if $m = 1$. Moreover, the approximation error is estimated by the L_2-modulus of continuity, which is less than the classical modulus of continuity in the uniform metric.

3. Table of practical results

In this section we present the results obtained by program which implements the previously mentioned algorithm. The program integrated the function $f(x) = sin(1/x)$ which is strongly oscillating. So, the classical cubature formulas and algorithms of numerical integration would be ineffective applied to this function.

Таблица 1. The average approximation error depending on n and m

n	m	Average approximation error
100	100	10^{-3}
100	1000	$4 \cdot 10^{-4}$
100	10000	$1.5 \cdot 10^{-4}$
100	100000	$5.3 \cdot 10^{-5}$
1000	100	$2 \cdot 10^{-4}$
1000	1000	$6 \cdot 10^{-5}$
1000	10000	$2 \cdot 10^{-5}$
1000	100000	$5.5 \cdot 10^{-6}$
10000	100	$4.5 \cdot 10^{-5}$
10000	1000	$1.5 \cdot 10^{-5}$
10000	10000	$4 \cdot 10^{-6}$
100000	100	$8.5 \cdot 10^{-6}$
100000	1000	$2 \cdot 10^{-6}$

Alas, in virtue of the computing capabilities of computer used these are the maximum values that could be obtained during the reasonable time. However, with the aid of the modern computers one can easily integrate strongly oscillating functions with minor approximation errors.

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